Extra Practice Problems 1

Here's a set of practice problems you can work through to help prepare for the midterm. We'll release solutions to these problems on Friday. If you have any questions about them, please feel free to stop by office hours!

First-Order Logic

Here's some more practice problems to help you get used to translating statements into first-order logic.

i. Given the predicates

   Person(p), which states that p is a person, and

   ParentOf(p_1, p_2), which states that p_1 is the parent of p_2,

   write a statement in first-order logic that says "someone is their own grandparent." (Paraphrased from an old novelty song.)

ii. Given the predicates

   Natural(n), which states that n is a natural number, and

   Integer(n), which states that n is an integer,

   along with the function symbol $f(n)$, which represents some particular function f, write a statement in first-order logic that says "$f : \mathbb{N} \to \mathbb{Z}$ is a bijection."

Induction and Set Theory

A set S is called an ***inductive set*** if the follow two properties are true about S:

- $0 \in S$.
- For any number $x \in S$, the number $x + 1$ is also an element of S.

This question asks you to explore various properties of inductive sets.

i. Find two different examples of inductive sets.

ii. Prove that the intersection of any two inductive sets is also an inductive set.

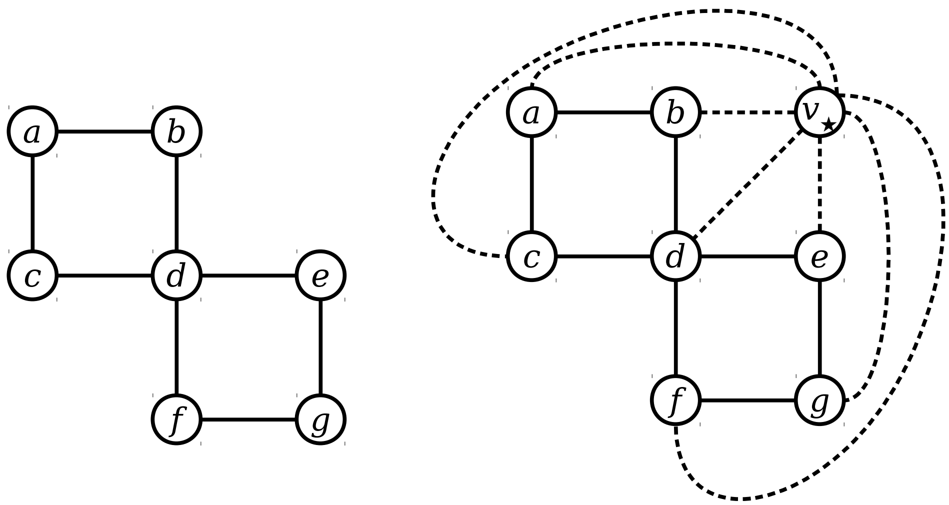
iii. Prove that if S is an inductive set, then $\mathbb{N} \subseteq S$.

iv. Prove that $\mathbb{N}$ is the *only* inductive set that's a subset of all inductive sets. This proves that $\mathbb{N}$ is, in a sense, the most "fundamental" inductive set. In fact, in foundational mathematics, the set $\mathbb{N}$ is sometimes defined as the one inductive set that's a subset of all inductive sets.

Graph Theory

In this question, you'll see a class of graphs called the ***outerplanar graphs*** that are closely related to the planar graphs. You'll then explore the connections between planar graphs and the outerplanar graphs.

Let's begin by introducing a new operation on graphs called ***augmentation***. If G is a graph, the augmentation of graph G, denoted Aug(G), is formed by adding a new node $v_\star$ to G, then adding edges from $v_\star$ to each other node in G. For example, below is a graph G and its augmentation Aug(G). To make it easier to see the changes between G and Aug(G), we've drawn the edges added in Aug(G) using dashed lines:



Graph G Aug(G)

Now, we can define the outerplanar graphs. An undirected graph G is called an ***outerplanar graph*** if Aug(G) is a planar graph. In other words, if Aug(G) is a *planar* graph, then the original graph G is an *outerplanar* graph.

i. Prove that every outerplanar graph is also a planar graph.

Recall that a ***k-coloring*** of a graph is a way of coloring each of the nodes in that graph one of k different colors so that no two nodes connected by an edge are given the same color.

The ***four-color theorem*** says that every planar graph is 4-colorable. This question asks you to prove an analogous result for outerplanar graphs.

ii. Prove the ***three-color theorem:*** every outerplanar graph is 3-colorable.